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Combinatorial Thinking to Solve the Problems of Combinatorics in Selection Type

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Abstract. Helping students solve combinatorics problems is an essential effort to solve a problem. Formulating the stages of combinatorial thinking is one of the means to help students solve the problems of combinatorics in selection type. The research paper discusses combinatorial thinking stages. It aims to formulate and describe the combinatorial thinking stage to solve combinatorics problems in selection type. The study used a qualitative approach. Combinatorial thinking stages include (1) giving one combinatorial question in selection type, (2) observing by recording the subjects when they answered it, (3) formulating combinatorial thinking stages based on the video-recording results and answer sheets, (4) conducting a triangulation, and (5) making a conclusion of combinatorial thinking stages. The results of the research show that there are four combinatorial thinking stages, such as identifying, selecting, concluding, and reflecting. Identifying is when the students can identify a problem by writing all the information inside the test instruments. Selecting happen when the students can choose the object, and then structure it based on the criteria of the test instruments. Concluding means that they have made a conclusion based on the criteria of the problem inside the test instruments. Finally, reflecting means that they have checked the objects selected and structured them well using a combinatorics concept and procedure.

Keywords: combinatorial thinking stages; combinatorics solved problem; combinatorics in selection type.

1. Introduction

The National Council of Teachers of Mathematics (NCTM) (2000) reported that there are five standards of the learning process of mathematics: *representations*, *reasoning and proof*, *communication*, *connections*, and *problem solving*. These standards are included in Higher Order Thinking Skills (HOTS). According to Resnick (1987), the characteristics of HOTS include being non-algorithmic, and complex, having different solutions, involving various judgments and interpretations, reflecting various criteria, and being more creative.

Heong, et al., (2012) proposed that the HOTS is very necessary to help students creatively express ideas for solving a problem. For this, it is hoped that students can understand the five standards for mathematical learning. Thus, it can be stated that learning mathematics is developing not only a counting skill but also a thinking skill in loving a mathematical problem. Solving the problem can not only be a routine test or question, but it also needs to put much emphasis on a daily question or literacy in mathematics. The PISA 2012 reported that literacy in mathematics is an individual's skill in formulating, applying, and interpreting mathematics in a variety of contexts. This skill is related to a thinking aspect or logical thinking (OECD, 2014). The era of Industrial Revolution 4.0 needs a logical thinking skill (Forsström & Kaufmann, 2018).

The NCTM (2000) suggested that combinatorial thinking is an essential element in comparison with other types of logical thinking and its existence can not be separated from mathematical learning. According to Graumann and Germany (2002), combinatorial thinking is a skill in solving a problem such as in geometry. Students must be in combinatorial thinking and must find a system to ensure that all alternatives have been discussed or related in a variety of patterns. In addition to geometry, combinatorial thinking is a skill in solving a problem such as statistics, algebra, and arithmetic (Batanero, et al., 1997). Therefore, combinatorial thinking is an essential skill for students before learning geometry, statistics, algebra, and arithmetic.

Tsai and Chang (2009) suggested that combinatorial thinking encourages students to be more creative, curious, and self-confident in solving a question. It is a basic skill that must be developed to build a potency and skill in critical thinking. It can encourage students to solve a mathematical problem. Rezaie dan Gooya (2011) stated that combinatorial thinking is a way of thinking in the concept of combinatorial learning.

Lockwood (2013) suggested that combinatorics should be included in the curriculum of mathematics education from primary school to higher education. Combinatorics includes structures with mathematic principles. Likewise, this is in studies of probability, computation, and enumeration learning, so it takes an important role in mathematics curriculum. In primary school, it can be used for developing students' thinking skill (English, 2005). Howefer, along with students' systematic advancement and development, they tend to have difficulties because of increasingly complex counting problems (Kavousian, 2008). The problems of combinatorics include selection, distribution, partition, repetition, and structure (Godino, et al., 2005). English (2005) proposed that combinatorics problems are differentiated into two types. One problem is related to a counting principle and uses a tree diagram, list, and table. The other problem is related to combinatorics types of selection, distribution, and partition.

Some researches studies have been conducted on strategies for combinatorics problem solving (Pizlo and Li, 2005; Melusova and Vidermanova, 2015). Pizlo and Li (2005) analyzed the combinatorics problem using a 15-puzzle strategy. Melusova and Vidermanova (2015) examined the difference in combinatorics problem solving before and after, formulated by a given strategy. The other researches aimed at combinatorics solving problem (Lockwood, 2013; Godino, et al., 2005; English, 2005). Lockwood (2013) formulated a combinatorial thinking model, including formulas/expressions, counting processes, and sets of outcomes. Formulas/expressions refer to mathematical expressions that yield some numerical value. The formulas/expressions can have combinatorics meaning (such as binomial $\binom{8}{3}$) or numeric operation combination (such as the amount of 9x13+3x12). Counting processes refer to the enumeration process (or series of processes) in which a counter engages (either mentally or physically) as they solve a counting problem. Sets of outcomes refer to the collection of objects being counted-those sets of elements that one can imagine being generated or enumerated by a counting process. The research by Godino, et al (2005) analyzed the students' answers in relation to a combinatorics-based solving of semiotics. English (2005) developed a combinatorics analysis using a meaningful learning.

According to Batanero, et al (1997), making combinatorial learning easy can formulate students' combinatorial thinking stages to solve a combinatorics problem. The stages are said to be necessary to formulate so they can help the students in combinatorics problem solving. In relation to Polya's stages, the combinatorial stages cover understanding a problem, solving a problem, realizing a plan, and reflecting (Polya, 1957). The stages are useful for solving a mathematical problem in general while the research paper discusses a combinatorics problem in particular. The combinatorics problem of the paper focus on discussing the combinatorics problem of selection type related to real life. According to Pourdavood and Liu (2017), understanding mathematics is said to be easier if it is related to a real context. The characteristics of the combinatorics problem solving of selection type include the keywords *select, take, draw, gather, pick,* etc. (Godino, et al., 2005).

The present research paper focuses on discussing the combinatorial thinking stages to solve the problems of combinatorics in selection type. The type must be discussed because it is a fundamental type before discussing partition and distribution problems. The objective of the study is to formulate and describe combinatorial thinking stages to solve combinatorics problem in selection type.

The research paper aims to contribute to combinatorial thinking development by (1) formulating combinatorial thinking stages in selection type, (2) identifying a person's combinatorial thinking when faced with a combinatorial problem in selection type, and (3) identifying the kinds of combinatorics problem solving in selection type.

2. Review of Literature

Rezaie and Gooya (2011) stated that combinatorial thinking means thinking specifically in the combinatorics learning concept. Combinatorics is a substantial part of mathematics. There are some kinds of combinatorics problems (Godino, et al., 2005; English, 2005). Godino (2005) proposed that combinatorics problems consist of selection, distribution, partition, and combination of distribution and partition. According to English (2005), the combinatorics problems comprise (1) a question of counting principles using a tree diagram, table, systematic list, and table and (2) combinatorial configurations, including (a) selection, (b) distribution, and (c) partition. Tucker (2012) suggested that counting principle (addition and multiplying) is a fundamental thinking in combinatorics problem solving.

Tucker (2012) proposed that combinatorics is a part of science that studies thestructure, operation, and selection in a discrete or finite system. According to English (2005), solving combinatorics problem uses various strategies from random selection to systematic selection type as well as from object repetition to all the combinations that may have been built.

3. Research Method

The research employed a qualitative approach. It focused on combinatorial thinking stages of the semester-two students of the Elementary School Teacher Education Department, Faculty of Teacher and Training, Universitas Muhammadiyah Surakarta (UMS), Provincial Jawa Tengah, Indonesia.

4. Research Design

The research employed a grounded theory since it is suitable to the characteristics of theoretical sampling, the constant-comparative method, and specific ways of coding (Lichtman, 2009: 73). Theoretical sampling involves collecting, analyzing, and determining the data gathered to produce combinatorial thinking stages and combinatorics problems. Constant-comparative method includes comparing one event with another event, one event with a category, and one category with another category.

The grounded theory emphasizes the data coding. Data coding covers (1) open coding, (2) axial coding, and (3) selective coding. Open coding means to read, understand literatures, and gather the amount of the categories relevant to combinatorial thinking. 2) Axial coding means formulating combinatorial thinking assumptions. 3) Selective coding means structuring combinatorial thinking predictors to combinatorics problem solving in selection type.

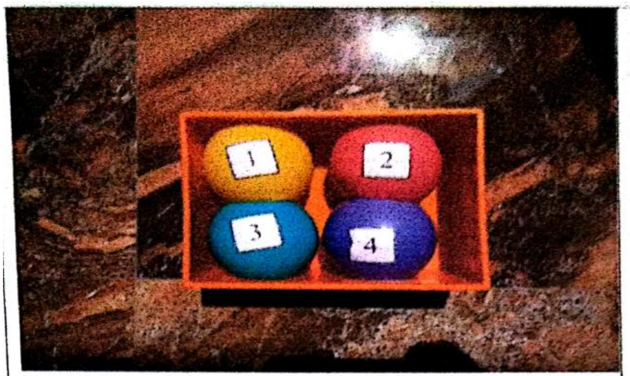
5. Subjects of Research

The research subjects were the students of the Elementary School Teacher Education, Faculty of Teacher and Training Education, Universitas Muhammadiyah Surakarta. The subjects totaled to 50 students. With the subjects, the researchers explored the information from different sources. These sources were useful for formulating the combinatorial thinking stages and finding the indicators of how the students used a combinatorial thinking to solve the problems of combinatorics in selection type. Out of 50 students, 20 people served a research subject by applying a purposive sampling method in random. They had taken the Elementary Basic Concept subject or course. In addition, the subject referred to the students' approval to be a research subject. Before the subjects answered the problems of combinatorics in selection type, the researchers contacted them.

6. Research Procedure

Combinatorial thinking included five stages. The first stage, the test instruments related to the problems of combinatorics in selection type were distributed to the students for them to solve. The use of the test instruments can be seen in Figure 1.

In Indonesian:



Chacha mempunyai kotak yang berisi empat bola dan diberi label angka 1, 2, 3, 4. Satu bola diambil dari kotak dan dicatat angka yang muncul pada bola tersebut. Kemudian bola tersebut dimasukkan kedalam kotak lagi. Proses tersebut diulangi hingga diperoleh tiga digit angka.

1. Sebutkan kemungkinan susunan tiga digit angka yang terbentuk!
2. Berapa banyak susunan digit yang dapat terbentuk?

Figure 1: Combinatorics problems in selection type

In English:

Chacha has a box of four balls, and they were inscribed with numbers 1, 2, 3, and 4. One was taken from the box, and the recording number was inscribed on it. After that, it was returned. This process was repeated three times to obtain a three-digit numeric.

1. Mention a three-digit number structure!
2. How many digit structures did you find?

In the second stage, the researchers observed, wrote, and recorded all the students' activities when they solved the problems of combinatorics in selection type. In the third stage, the researchers analyzed the students' combinatorial thinking stages based on observation and recording of the results. The analysis results were in relation to a conclusion of the stages and indicators of the research subjects.

In the fourth stage, the researcher applied a data triangulation technique to confirm the analysis results by in-depth interview. The interview was in semi-structure so that there was no interview guidance. It was dependent on the individual research subjects.

In the final stage, the researchers made a conclusion of combinatorial thinking based on observation, recording, and interview. Thus, data of the students' combinatorial thinking stages and attitude indicators in solving a combinatorics problem of selection type were recorded. Table 1 reports the combinatorial thinking stages.

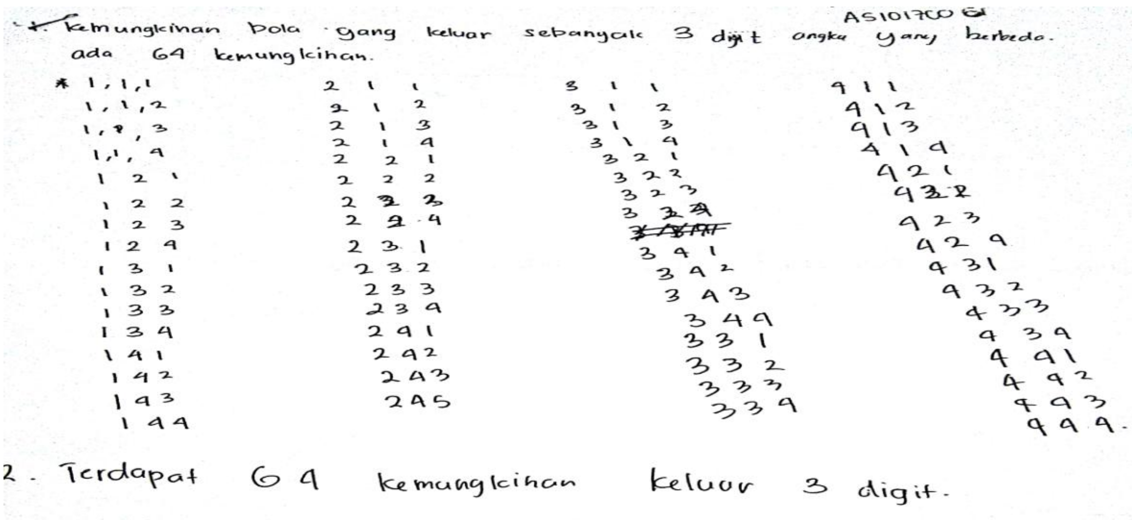
Table 1: Students' stages of combinatorial thinking in solving the problems of combinatorics in selection type

Stages	Descriptions
Identifying	1. Understanding the sets 2. Identifying an object as the sets 3. Understanding that all the notation sets had to be chosen based on the problem criteria
Selecting	1. Selecting the object structures based on the criteria 2. Structuring the probable objects based on the criteria
Concluding	1. Making a conclusion based on the problem
Reflecting	1. Checking all the probable structures systematically 2. Checking all the probable structures by employing combinatorics concept and procedure

7. Analysis Results and Discussion

The researchers provided the combinatorics in selection type to the research subjects. Five subjects or students expressed the combinatorial thinking stages continuously. The samples of the students' answers are reported in Figure 2.

In Indonesian:



1. Kemungkinan pola yang keluar sebanyak 3 digit angka yang berbeda. ada 64 kemungkinan.

* 1, 1, 1	2 1 1	3 1 1	4 1 1
1, 1, 2	2 1 2	3 1 2	4 1 2
1, 1, 3	2 1 3	3 1 3	4 1 3
1, 1, 4	2 1 4	3 1 4	4 1 4
1 2 1	2 2 1	3 2 1	4 2 1
1 2 2	2 2 2	3 2 2	4 2 2
1 2 3	2 2 3	3 2 3	4 2 3
1 2 4	2 2 4	3 2 4	4 2 4
1 3 1	2 3 1	3 4 1	4 3 1
1 3 2	2 3 2	3 4 2	4 3 2
1 3 3	2 3 3	3 4 3	4 3 3
1 3 4	2 3 4	3 4 4	4 3 4
1 4 1	2 4 1	3 3 1	4 4 1
1 4 2	2 4 2	3 3 2	4 4 2
1 4 3	2 4 3	3 3 3	4 4 3
1 4 4	2 4 5	3 3 4	4 4 4.

2. Terdapat 64 kemungkinan keluar 3 digit.

In English:

1. It is probable that the balls out of the box amount to three different or similar digits

1 1 1	2 1 1	3 1 1	4 1 1
1 1 2	2 1 2	3 1 2	4 1 2
1 1 3	2 1 3	3 1 3	4 1 3
1 1 4	2 1 4	3 1 4	4 1 4
1 2 1	2 2 1	3 2 1	4 2 1
1 2 2	2 2 2	3 2 2	4 2 2
1 2 3	2 2 3	3 2 3	4 2 3
1 2 4	2 2 4	3 2 4	4 2 4
1 3 1	2 3 1	3 4 1	4 3 1
1 3 2	2 3 2	3 4 2	4 3 2
1 3 3	2 3 3	3 4 3	4 3 3
1 3 4	2 3 4	3 4 4	4 3 4
1 4 1	2 4 1	3 3 1	4 4 1
1 4 2	2 4 2	3 3 2	4 4 2
1 4 3	2 4 3	3 3 3	4 4 3
1 4 4	2 4 4	3 3 4	4 4 4

2. Thus, it is probable that there are 64 in three different or similar digits.

Figure 2: **One sample out of the five subjects' answer**

Based on the S' (Subject) answer, the subject could select object structures based on the criteria and structured objects that might be based on the criteria. It is indicated that the subject could answer all the probabilities. The subject had been able to solve all the probable structures by listing one by one a formed three-digit number structure. It is described in the following interview.

R (Researcher): What did you think about this test or question?

S (Subject): In our opinion, there was a child who had one box of four balls. Each was inscribed with one numeric. One by one, the ball was taken, and the numeric was recorded. Then, it was returned again into the box. These processes occurred three times. Those were to answer the questions of items '1' and '2'. ***(identifying)***

R: How did you answer the questions? What did you mean by the numbers '1', '2', and '3'?(indicated in the S1' answer sheet)

S: I made a list, ma'am. For taking the first, I began from number 1 for taking the first ball, number 2 for taking the second ball, to number 3 for taking the third ball to number 4 for taking the fourth ball. For taking the second, it was the same as the first. I began from taking numbers 1, 2, 3 to 4. For taking the third, I began from numbers 1, 2, 3 to 4, then from numbers 1, 2, 3 to 4 in sequence. Numbers 1, 1, 1 were the numeric 1. It was the first digit from taking the first ball. The next digit 1 was the second digit from taking the second ball. Then, the next digit 1 was the third digit from taking the third ball. ***(selecting and concluding)***

R: How did you think that you had argued all of the solutions to the problems?

S: I had checked with the formula, ma'am. ***(reflecting)***

The results of the research by Lockwood (2013) formulated a combinatorial thinking model. In the results, Godino, et al., (2005) analyzed a solution or answer based on the types of selection, distribution, and partition problems, as well as the combinations of distribution and partition problems in semiotics. The present research paper developed the two research results by formulating the stages of combinatorial thinking in solving the problems of combinatorics in selection type. The combinatorial thinking model was used for formulating combinatorial thinking stages. Combinatorics problem in selection type was employed since the type was the most fundamental among the other types. It was also relevant to Polya (1957), stating the combinatorial thinking stages include understanding a problem, planning to solve a problem, doing a plan, and reflecting. These stages are useful for mathematics problem solving in general while combinatorial thinking stages are helpful for solving combinatorics problem in selection type. Combinatorial thinking for solving the problems of combinatorics in selection type begins with identifying a problem, selecting object structures, concluding, and reflecting. Reflecting involves ensuring whether the solutions or answers are true or not. The stages of combinatorial thinking to solve the problems of combinatorics in selection type can be seen in Figure 3.

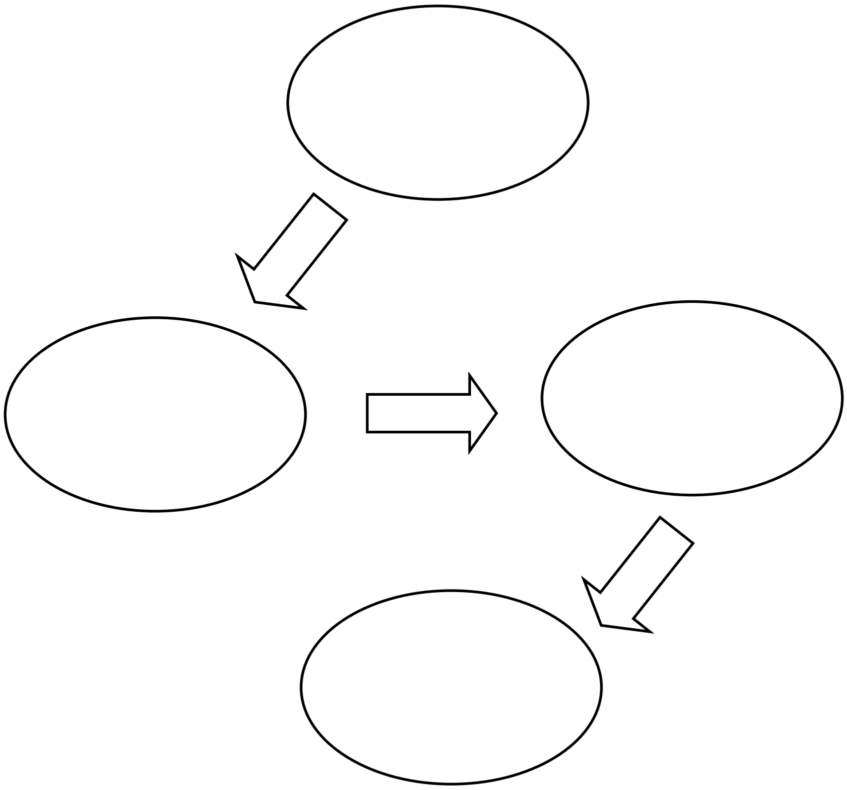


Figure 3: Stages of combinatorial thinking to solve the problems of combinatorics selection type

A theory can be constructed with two methods: inductive and deductive. The research used the deductive method. The researchers formulated some hypothesis of combinatorial thinking by referring to relevant theories. Based on the hypothesis, they formulated the predictors of combinatorial thinking stages to solve the problems of combinatorics in selection type. The indicators of the stages of combinatorial thinking can be seen in Table 2.

Table 2: Indicators of the stages of combinatorial thinking to solve the problems of combinatorics selection type

Combinatorial Thinking Stages	Indicators
Identifying	1. Identifying the objects as sets 2. Understanding that all of the members of the sets would be selected based on the problem criteria
Selecting	1. Selecting the object structures based on the criteria 2. Structuring the probable objects based on the criteria
Concluding	Making a conclusion based on the problems
Reflecting	1. Checking all the probable structures systematically 2. Checking all the probable structures by employing the combinatorics concept and procedure

As described in Table 2, in the identifying stage, the subjects could write or express information in the instrument, and identify an object as a sets and all the notation sets had to be selected based on the combinatorial criteria of selection type. As proposed by Godino (2005), combinatorial problems should include selection, distribution, partition, and combination of distribution and partition.

In the selecting stage, the subjects indicated the answers in a diagram, scheme, list, and table. They selected the object structure based on the criteria and structured the object perhaps based on the criteria. The structure was represented with a table, diagram, list, and scheme (Godino, et al., 2005). According to English (2005), the use of a tree diagram, systematic list, and table is a basic procedure for solving a combinatorics problem. In the concluding stage, the subjects could make a conclusion accurately based on the problem. The implication of the stage was related to the success in solving a problem (Eizenberg & Zaslavsky, 2004).

Finally, in the reflecting stage, the subject could check all the structures systematically and apply a concept and procedure of combinatorial problem. It is relevant to the research result by Batanero, et al. (1997). After a formal-operational stage period, the adolescences should be able to find a procedure of combinatorics construction systematically. Similarly, the present research paper used the students of Elementary School Teacher Education. They were in the formal-operational stage.

8. Conclusion

Based on the research result, it can be concluded that there are four components of combinatorial thinking to solve the problems of combinatorics in selection type. These components encompass identifying, selecting, concluding, and reflecting. *First*, identifying means that the students can recognize a combinatorics problem of selection type. They can identify an object as a question of the sets. Then, the notation sets must be selected based on the problem criteria. *Second*, selecting is a main characteristic of problems of combinatorics in selection type. In this stage, the students can choose the object structure based on the problem criteria and they can structure the object.

Third, concluding means that the students can make a conclusion based on the object structure as determined. *Fourth*, reflecting means that the students can recheck their answers or solutions procedurally.

It is hoped that the research results of combinatorial thinking components to solve the problems of combinatorics in selection type can contribute to the students learning. They can use these components to solve the problems of combinatorics in selection type. It is suggested that in further studies, the research results can be used as a reference to other relevant researches although in different subjects. In addition, the combinatorial thinking stages can help students who are having difficulty in solving a combinatorics problem. So, it is recommended that a further research will be able to formulate combinatorial thinking stages for combinatorics problems in selection type.

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